CORRECTIONS IN THE TEXT

 For heat Transfer

terms for the q energy fluxes (i.e., q_x, q_y, q_z, etc.). These expressions are given below for the rectangular case:

$$q_x = -k\frac{\partial T}{\partial x}, \qquad \overset{q_y}{\cancel{q_x}} = -k\frac{\partial T}{\partial y}, \qquad q_z = -k\frac{\partial T}{\partial z} \tag{5-5}$$

the cylindrical case:

$$q_r = -k\frac{\partial T}{\cancel{\partial \theta}\,\partial r}, \qquad q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}, \qquad q_z = -k\frac{\partial T}{\partial z} \tag{5-6}$$

and the spherical case:

$$q_r = -k\frac{\partial T}{\partial r}, \qquad q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta}, \qquad q_\phi = -k\frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi} \tag{5-7}$$

Equations (5-8), (5-9), and (5-10) give the revised forms of equations (5-2) through (5-4). The rectangular coordinate equation is (5-8), while (5-9) and (5-10) are, respectively, for the cylindrical and spherical cases.

$$\underbrace{\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right)}_{\text{Convection}} = \underbrace{k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]}_{\text{Heat conduction}}$$

$$\underbrace{+ 2\mu\left\{\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right\}}_{\text{Viscous dissipation}} \tag{5-8}$$

Then, for cylindrical coordinates we have

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$+ 2\mu\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left[\frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\right]^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right\} + \mu\left\{\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)^2\right.$$

$$\left. + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right)^2 + \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\right\} \tag{5-9}$$

and for spherical coordinates we have

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right) = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right.$$

$$\left. + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

If we have a cylinder that has only radial conduction, then

$$q^1 = -kA\frac{dT}{dr} \tag{5-14}$$

Now because $A = 2\pi rL$ for the cylinder surface, we obtain for a hollow cylinder (inner radius r_i and outer radius r_o) a q^1 as shown:

$$q^1 = \frac{2\pi kL(T_i - T_o)}{\ln(r_o/r_i)} \tag{5-15}$$

For multiple cylindrical sections we have

$$q^1 = \frac{2\pi L(\Delta T)_{\text{overall}}}{\dfrac{\ln(r_2/r_1)}{k_a} + \dfrac{\ln(r_3/r_2)}{k_b} + \dfrac{\ln(r_4/r_3)}{k_c}} \tag{5-16}$$

GENERAL CASE FOR STATIC SYSTEMS

The most general case for the static system is one in which there is the possibility of heat generation and unsteady-state heat transfer. For such a case where all velocities are zero by combining equations (5-2) and (5-5) we obtain

$$\rho C_P\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{k\partial T}{\partial z}\right) + \dot{q} \tag{5-17}$$

The q term is for heat generation (i.e., heat of reaction, latent heat, etc.).

For a steady-state ($\partial T/\partial t = 0$) one-dimensional slab with a k independent of position, equation (5-17) becomes

$$\frac{\partial^2 T}{\cancel{\partial X^2}} + \frac{\dot{q}}{k} = 0 \tag{5-18}$$

∂x^2 ← lower case

which is the basic equation for one-dimensional slab conduction with heat generation.

The corresponding cylindrical coordinate equation to (5-17) for only radial conduction is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \underbrace{\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}}_{\text{No }\theta\text{ conduction}} + \underbrace{\frac{\partial^2 T}{\partial z^2}}_{\text{No }z\text{ conduction}} + \frac{\dot{q}}{k} = \frac{\rho C\rho}{k}\frac{\partial T}{\partial t} \tag{5-19}$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\dot{q}}{k} \tag{5-20}$$

Again this represents the basic equation for a cylinder with heat generation and radial conduction.

In unsteady-state cases, temperature change with time must be considered. Hence, for a system in rectangular coordinates without heat generation we obtain

$$\rho C_P \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k \partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k \partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{k \partial T}{\partial z}\right) \tag{5-21}$$

Note that the solution of equation (5-21) is complex even for the one-dimensional case because two partial differentials are involved. Fortunately, a great deal of work has been done in this area. The results of these efforts are tabulated in textbooks such as those authored by Carslaw and Jaeger (1) and Crank (2).

In addition, sets of charts have been developed that related temperature to position and time for slabs, cylinders, and spheres (1, 3, 4); these are shown in Figures 5-3 through 5-5.

The terms in these charts are: ~~T_1~~ t_a, surface temperature; ~~T_0~~ t_b, temperature at a given point at zero time; T, the temperature at that point when a time, t, has elapsed; α, the materials' thermal diffusivity; x_1, a characteristic dimension (a radius, or half thickness); x, the position for T_0 and T; n, a dimensionless position (x/x_1); m, a dimensionless function. $\dfrac{k}{hx_1}$ (the ratio of k to h (a film coefficient) times x_1).

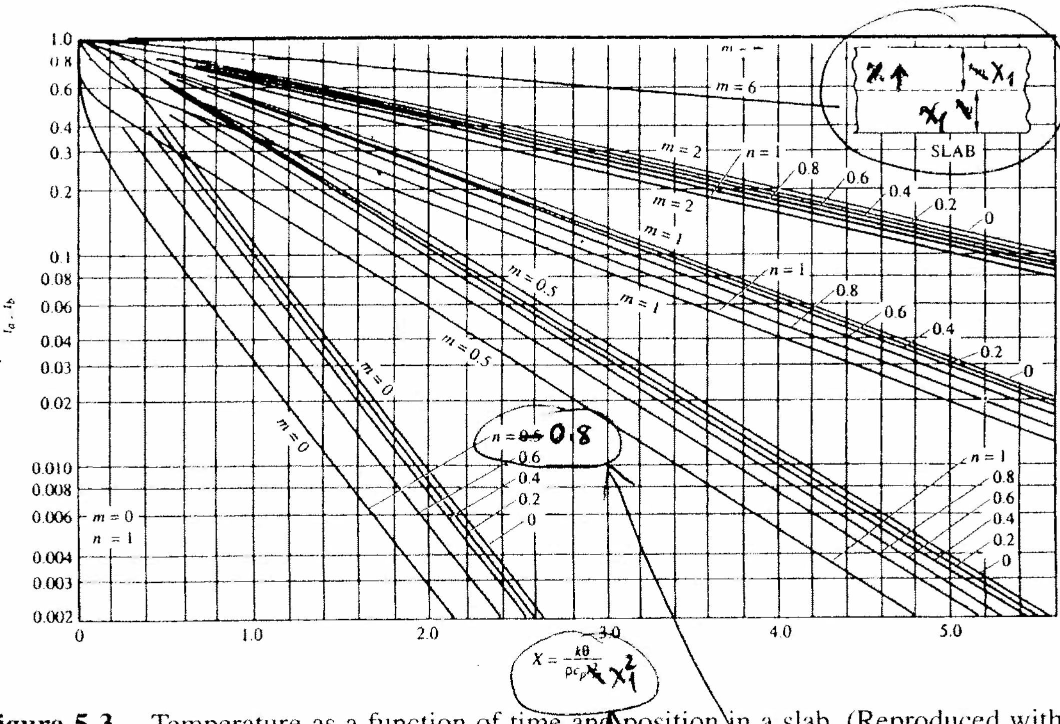

Figure 5-3. Temperature as a function of time and position in a slab. (Reproduced with permission from reference 3. Copyright 1923, American Chemical Society.)

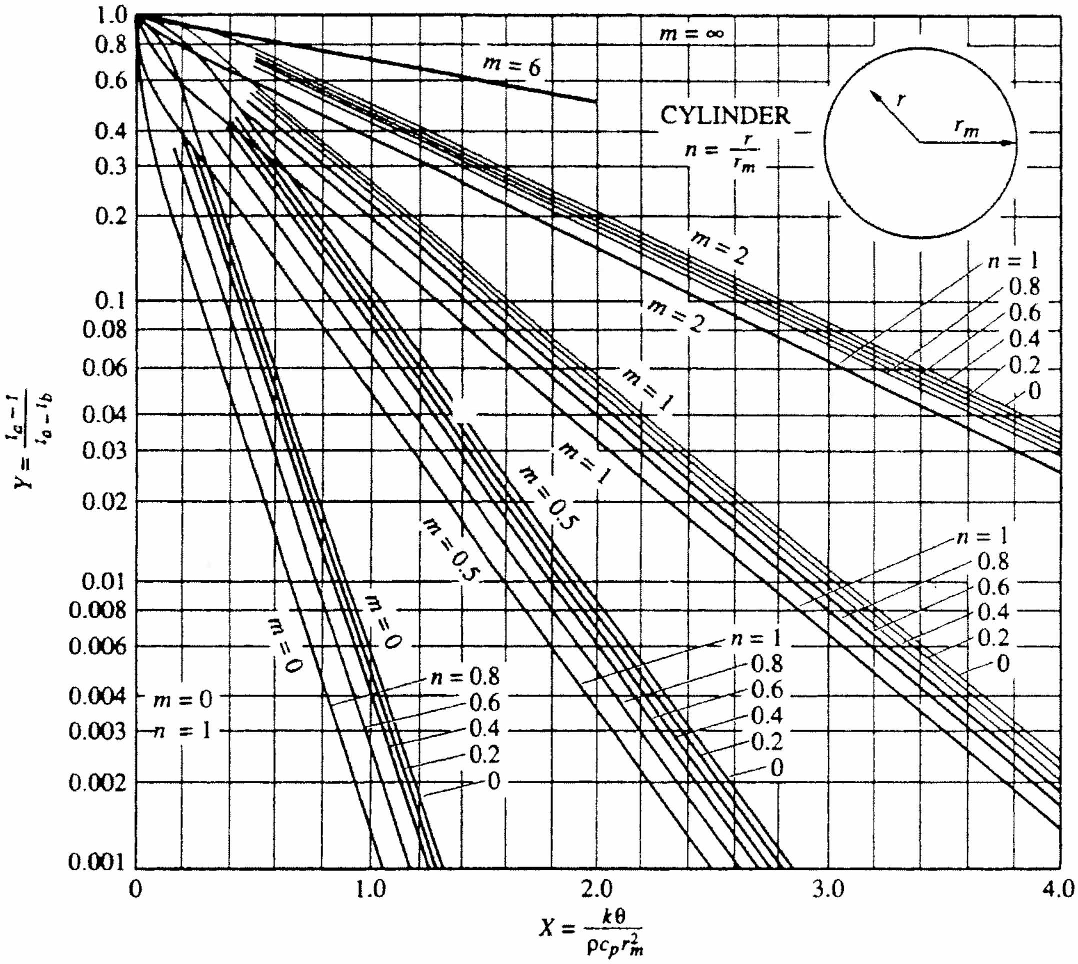

Figure 5-4. Temperature as a function of time and position in a long cylinder. (Reproduced with permission from reference 3. Copyright 1923, American Chemical Society.)

WORKED EXAMPLES

Example 5-1 A cold storage room has walls constructed of 0.102-m corkboard contained between double wooden walls each 0.0127 m thick. What is the rate of heat loss if wall surface temperature is $-12.2°C$ inside and $21.1°C$ outside? Also, what is the temperature at the interface between the outer wall and the corkboard?

Values of thermal conductivity for wood and corkboard are 0.1073 W/m °C and 0.0415 W/m °C, respectively.

Because this is a case of multiple slab resistances, we use equation (5-13).

$$q^1 = \frac{(\Delta T)_{\text{overall}}}{\dfrac{\Delta x_a}{k_a A} + \dfrac{\Delta x_b}{k_b A} + \dfrac{\Delta x_c}{k_c A}}$$

Using this ordinate and reading to the line for $m = 0, n = 0$ gives an abscissa of 1.13. Then because

$$1.13 = \frac{kt}{\rho C\rho x_1^2}$$

We solve for t and obtain a value of ~~612~~ 177 sec.

Example 5-5 Find the temperature profile for the laminar flow of a Newtonian fluid (constant density and thermal conductivity) if there is a constant energy flux at the wall.

Flow is along the tube axis (z direction). This means that the only velocity is $\cancel{V_z}$ v_z. Also, viscous dissipation is neglected. The proper equation in this case is (5-9), which upon evaluation

$$\rho\hat{C}_\rho\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

(Steady state) (No $\cancel{V_r}$ v_r, no $\cancel{V_\theta}$ v_θ) (No θ change) (z convection larger than z conduction)

becomes

$$\rho\hat{C}_\rho\left(v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right]$$

For the developed flow case (see Example 2-2) we have

$$\overset{v_z}{\cancel{V_z}} = (\overset{v_z}{\cancel{V_z}})_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Substituting for $\overset{v_z}{\cancel{V_z}}$ and using the following boundary conditions, we obtain

$$r = 0, \qquad \frac{\partial T}{\partial r} = 0$$

$$r = R, \qquad T = TW$$

$$0 \leq r \leq R, \qquad \frac{\partial T}{\partial z} = \text{constant}$$

The last condition results because the wall heat flux is constant and the average fluid temperature increases linearly with z.

Solving the energy equation gives

$$T_W - T = \frac{(\overset{v_z}{\cancel{V_z}})\max\rho C\rho}{16R^2k}\left(\frac{\partial T}{\partial z}\right)(3R^{\cancel{4}\,4} - 4r^2R^2 + r^4)$$

so that

$$-k\frac{\partial^2 T}{\partial x^2} = \mu\left(\frac{\partial v_z}{\partial x}\right)^2$$

The velocity term can be obtained from the Equation of Motion z component (2-27). From this equation (with pressure and gravitational effects both zero), we obtain

$$\frac{\partial^2 v_z}{\partial x^2} = 0$$

Integrating with the boundary conditions

$$v_z = V \text{ (i.e., } \Omega R), \qquad x = B$$
$$v_z = 0, \qquad x = 0$$

the relation

$$\frac{v_z}{V} = \frac{x}{B}$$

is determined.

Then

$$-k\frac{d^2 T}{dx^2} = \mu\frac{V^2}{B^2}$$

Solving with boundary conditions

$$T = T_0, \qquad x = 0$$
$$T = T_1, \qquad x = B$$

we obtain

$$T = T_0 + (T_1 - T_0)\frac{x}{B} + \frac{\mu V^2}{2\,k}\frac{x}{B}\left(1 - \frac{x}{B}\right)$$

or

$$\frac{T - T_0}{T_1 - T_0} = \frac{x}{B} + \frac{\mu V^2}{2k(T_1 - T_0)}\frac{x}{B}\left(1 - \frac{x}{B}\right)$$

The dimensionless grouping in the above expression is known as the Brinkman number (Br).

$$\text{Br} = \frac{\mu V^2}{k(T_1 - T_0)} = \frac{\text{Heat generated by viscous dissipation}}{\text{Conduction heat transfer}}$$

This group indicates the impact of viscous dissipation effects.

For the case at hand, $T_1 = T_0$ and

$$T - T_0 = \frac{\mu V^2}{2k}\frac{x}{B}\left(1 - \frac{x}{B}\right)$$